Lezione 15

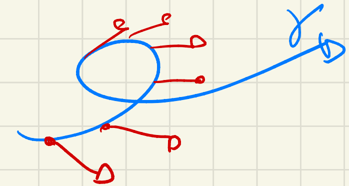
(M, g) varietà pseudo-Riemanniana

- distanza (solo nel caso Riemanniano)
- forma volume $\qquad \omega = \sqrt{|\det g_{ij}|}\, dx^1 \wedge \dots \wedge dx^n$
- g_{ij} prod. scalare $\Rightarrow \quad \|v\| = \sqrt{\underbrace{|g_{ij} v^i v^j|}_{g(v,v)}}$
- $L(\gamma)$ lunghezza di curva γ
- ∇ connessione (di Levi-Civita)

unica ∇ simmetrica e compatibile con g

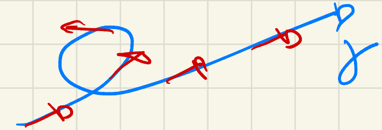
$\Gamma_{ij}^k = \Gamma_{ji}^k$

- trasporto parallelo



- geodetica

$$\ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k e^k = 0$$



- curvatura

Oss: $\ddot{x} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k e^k = 0 \qquad \ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0$

n equazioni differenziali al II ordine $\qquad k = 1, \dots, n$

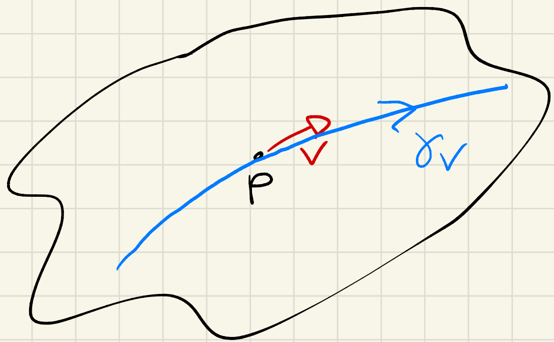
$$\ddot{x}^k(t) + \dot{x}^i(t)\, \dot{x}^j(t)\, \Gamma_{ij}^k(x(t)) = 0$$

NON È LINEARE $\Longrightarrow$ $x(t)$ può non esistere $\forall t \in \mathbb{R}$

$\exists!$ locale : $\exists!$ $x(t)$ se fisso $x(t_0)$, $\dot{x}(t_0)$

LOCALMENTE

Cor: (M, ∇) $\forall p \in M$, $\forall v \in T_pM$ $\exists!$ geodetica massimale

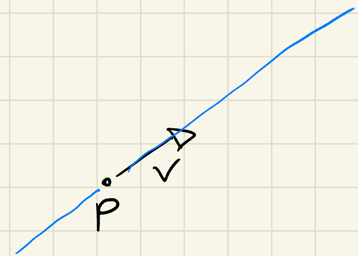


$\gamma_v : I_v \to M$ t.c. $I_v \subseteq \mathbb{R}$

$\gamma_v(0) = p$

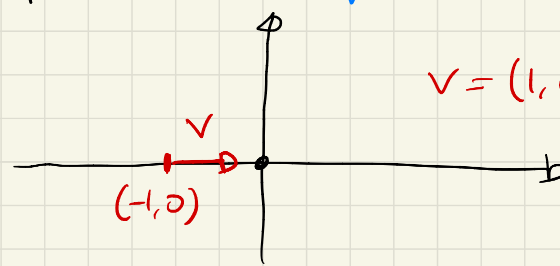
$\dot{\gamma}_v(0) = v$

Es: $M = \mathbb{R}^2$, ∇^E



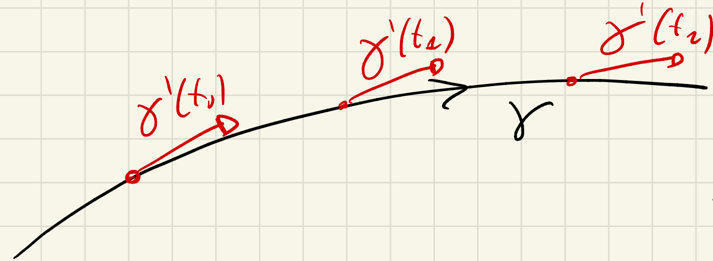
$\gamma_v(t) = p + tv$ $\quad I_v = \mathbb{R}$

$M = \mathbb{R}^2 \setminus \{0\}$, ∇^E



$v = (1,0)$

Oss: Se γ geodetica, allora $\|\gamma'(t)\| = \text{cost}$



γ' è parallelo

$\Downarrow$

$\gamma'(t)$ è ottenuto da $\gamma'(t_0)$ per trasporto parallelo lungo γ, che è isometria perché g e ∇ sono compatibili

Quindi ci sono 3 tipi di geodetiche:

TEMPO, SPAZIO E LUCE

(anche $\langle \gamma'(t), \gamma'(t) \rangle$ è costante)

Oss: Se $\gamma(t)$ è geodetica e $c > 0$ allora anche $\eta(t) = \gamma(ct)$

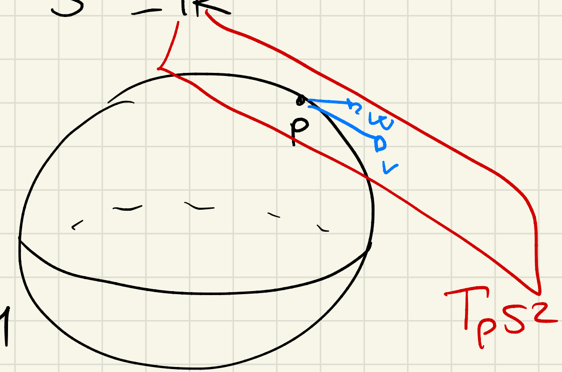
$$\underset{c^2}{\ddot{x}} + \underset{c \cdot c}{\dot{x}^i \dot{x}^j} \Gamma_{ij}^k e_k = 0$$

$x(t)$ è soluzione $\Rightarrow$ $x(ct)$ è soluz.

Sottovarietà pseudo-Riemanniana

$N \subseteq M$

Es: $S^2 \subseteq \mathbb{R}^3$



$(M, g) \dashrightarrow g^N$ su N

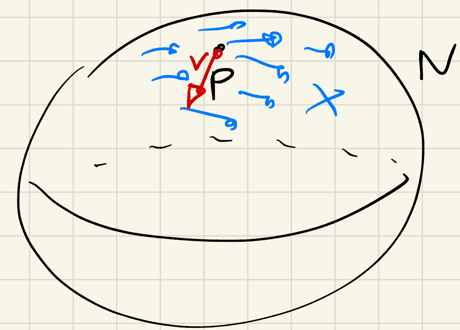
$$g^N(p) = g(p)\big|_{T_pN \subseteq T_pM}$$

Che relazione c'è tra ∇^N e ∇^M?

$$\begin{tikzcd}
(M, g) \arrow[r, dashed] \arrow[d, dashed] & \nabla^M \arrow[d, dashed, "?"] \\
(N, g^N) \arrow[r, dashed] & \nabla^N
\end{tikzcd}$$

Prop: $\nabla^N = \pi \circ \nabla^M$ cioè:

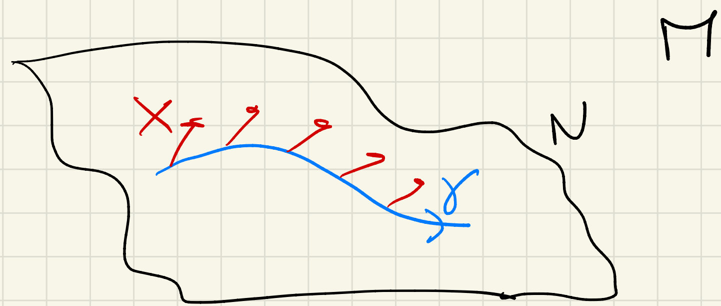
$$T_pN \ni \nabla^N_v X = \pi \circ \nabla^M_v X \in T_pM$$



$\pi : T_pM \to T_pN$

proiezione ortogonale

Cor:



X è parallelo rispetto a $N \overset{\text{def}}{\iff} \nabla^N_{\gamma'(t)} X = 0 \ \forall t$

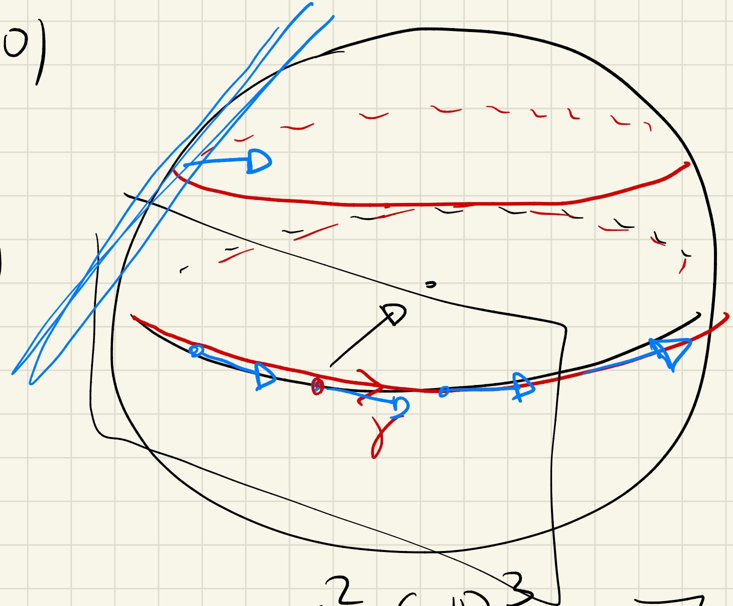
$\iff \nabla^M_{\gamma'(t)} X \perp T_{\gamma(t)}N \quad \forall t$

$\gamma''(t) = (-\cos t, -\text{sen}\, t, 0)$

$\gamma(t) = (\cos t, \text{sen}\, t, 0)$

$\gamma'(t) = (-\text{sen}\, t, \cos t, 0)$

γ è geodetica in S^2



perché vale il seguente

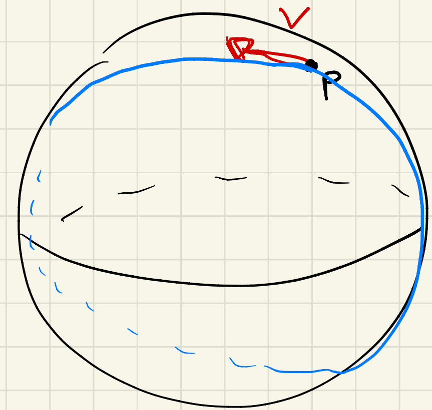
$S^2 \subseteq \mathbb{R}^3$

Cor: $N \subseteq M$ $\quad \gamma : I \to N$ è geodetica

$\iff \forall t \in I \quad \nabla^M_{\gamma'(t)} \gamma'(t) \perp T_{\gamma(t)} M$

Cor: $N \subseteq \mathbb{R}^3$ superficie $\gamma: I \to N$ curva

γ geod. $\Longleftrightarrow$ $\gamma''(t) \perp T_{\gamma(t)} M$



Es: $p \in S^2$ $\|p\| = 1$

$v \in T_p S^2 = p^\perp$

$\|v\| = 1$

$\gamma_v(t) = p \cos t + v \operatorname{sen} t$

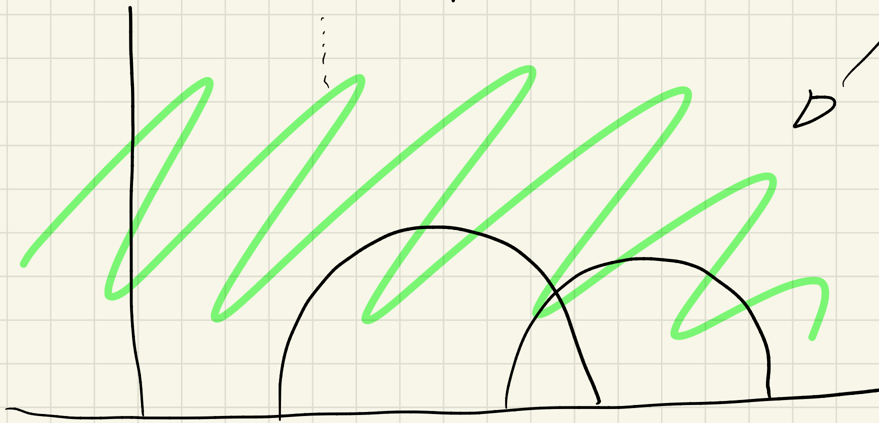
- $\gamma_v(t) \in S^2 \quad \forall t \qquad \|\gamma_v(t)\| = 1$
- $\ddot{\gamma}_v(t) = -\gamma_v(t) \perp (\gamma_v(t)^\perp)$

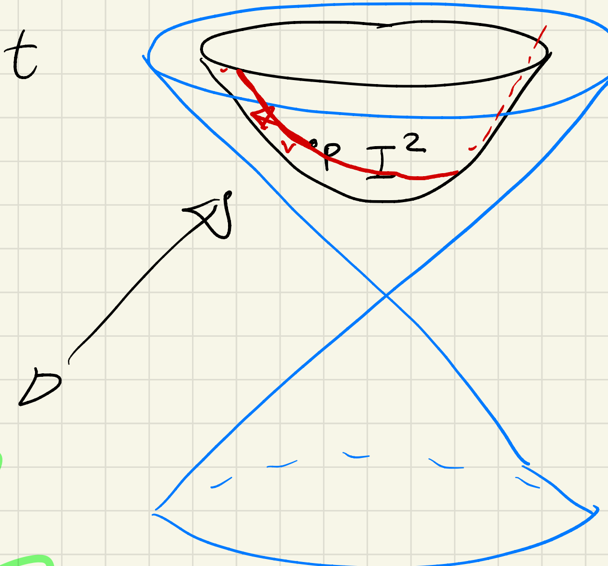
Es: $p \in I^2 \subseteq \mathbb{R}^{2,1}$ $v \in T_p I^2$ $\|v\| = 1$

$$\gamma_v(t) = p\cosh t + v \,\mathrm{senh}\, t$$

si ottiene in modo simile

$$T_p I^2 = p^{\perp}$$





CURVATURA

(M, ∇) $\qquad \nabla \dashrightarrow T$ TORSIONE $(1,2)$

$T \equiv 0$ (di solito) $\qquad T(p) = \nabla_X Y - \nabla_Y X - [X, Y]$

Def: TENSORE DI RIEMANN $(1,3)$ $\qquad T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$

(M, ∇)

$\forall p \in M, \quad \forall u, v, w \in T_pM$

$u, v, w \to X, Y, Z$ estensioni arbitrarie

$$R(p)(u, v, w) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$\nabla_v w$ non ha senso

$\nabla_u \boxed{\nabla_v Z}$ non ha senso

vettore

$\nabla_u \nabla_Y Z$ ha senso

Teo: Non dipende dalle estensione

In carte $\quad R_{ijk}{}^{l} = \dfrac{\partial \Gamma_{jk}^{l}}{\partial x^{i}} - \dfrac{\partial \Gamma_{ik}^{l}}{\partial x^{j}} + \Gamma_{im}^{l}\Gamma_{jk}^{m} - \Gamma_{jm}^{l}\Gamma_{ik}^{m}$

$\boxed{n^4}$

$$R(u,v,w)^{l} = R_{ijk}{}^{l} u^{i} v^{j} w^{k}$$

$$R(u,v,w) = R_{ijk}{}^{l} u^{i} v^{j} w^{k} e_{l}$$

Simmetrie: riducono i gradi di libertà da n^4 a ....

$n=2$: $2^4 \to 1$ $\quad R_{121}{}^{2}$ $\qquad$ $n=4$: $4^4 \to 20$

$n=3$: $3^4 \to 6$

$$X, Y \in \mathfrak{X}(M) \rightarrow \nabla_X Y \in \mathfrak{X}(M)$$

$$(\nabla_X Y)(p) := \nabla_{X(p)} Y$$

$T \equiv 0$ $\Rightarrow \nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$

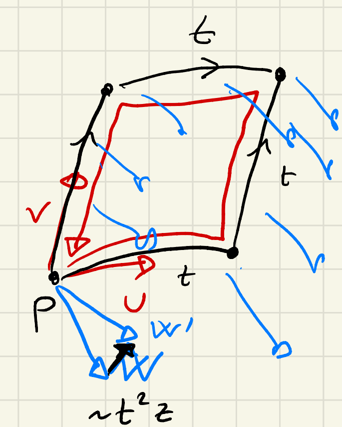
$$\Rightarrow \boxed{[X, Y] = \nabla_X Y - \nabla_Y X}$$

$$\nabla_V Y$$

INTERPRETAZIONE GEOMETRICA

$$u, v, w \longrightarrow z = R(p)(u, v, w)$$

$t > 0$



$$w' = w + t^2 z + o(t^2)$$

Tensore di Ricci

$$R_{ijk}{}^{\ell} \qquad \longrightarrow \qquad R_{ij} = R_{kij}{}^{K} \qquad \text{contrazione}$$

$$(1,3) \qquad\qquad (0,2)$$

$$A^i{}_j \longrightarrow A^i{}_i = \mathrm{tr} A$$

$$(1,1) \qquad\qquad (0,0)$$

Teo: $R_{ij} = R_{ji}$ è simmetrico

COMPARE NELL' EQ. DI CAMPO DI EINSTEIN

g_{ij} tensore metrico

T_{ij} tensore energia - impulso

CURVATURA SCALARE $\qquad R_{ij} \qquad R_i{}^j = R_{ik} g^{kj}$

$$R = R_i{}^i = R_{ik} g^{ki}$$

Riemanniana

$$B(p,r) = \{ q \in M \mid d(p,q) < r \}$$

$$\mathrm{Vol}(B(p,r)) = V_r^n \left(1 - \frac{1}{6(n+2)} R r^2 + o(r^2) \right)$$

Euclideo

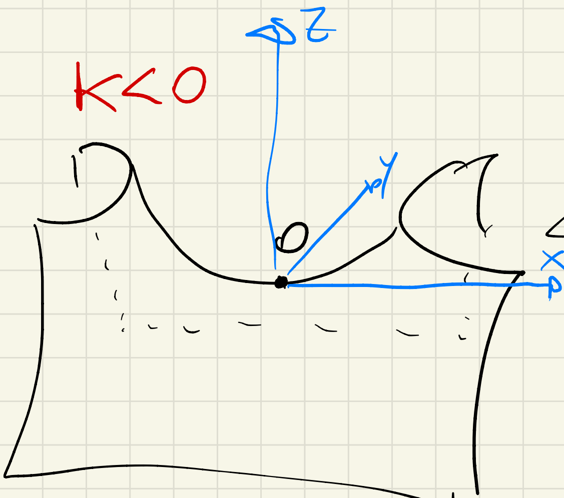
$B(N,r) < \pi r^2$

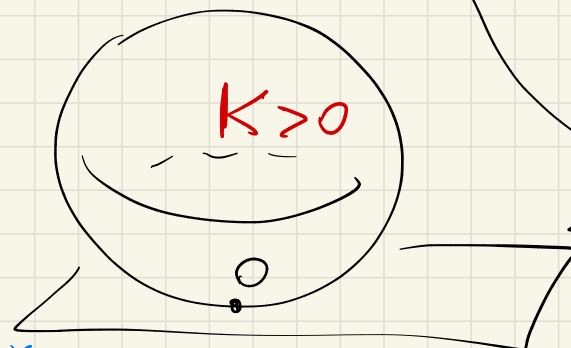
$R = 1$

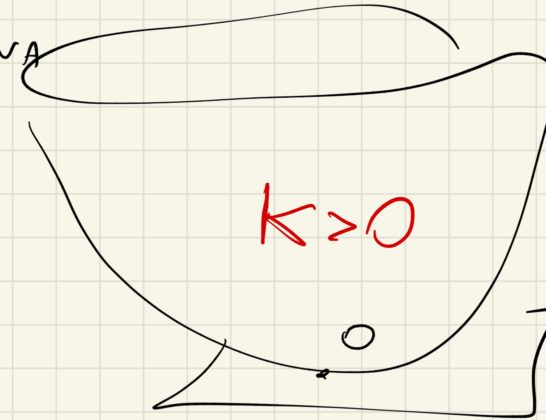
$S^n \quad R \equiv 1 \qquad \mathbb{R}^n \quad R \equiv 0 \qquad H^n \quad R \equiv -1$

CURVATURA GAUSSIANA

$S \subseteq \mathbb{R}^3$ superficie







$$z = x^2 - y^2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$z = \frac{1}{2}(x^2 + y^2) + o(\dots)$$

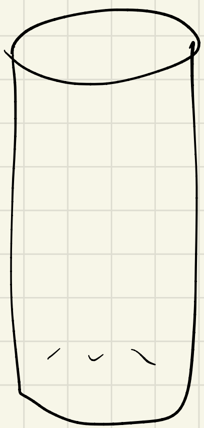
S è loc. grafico di f

$$\begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \lambda > 0$$

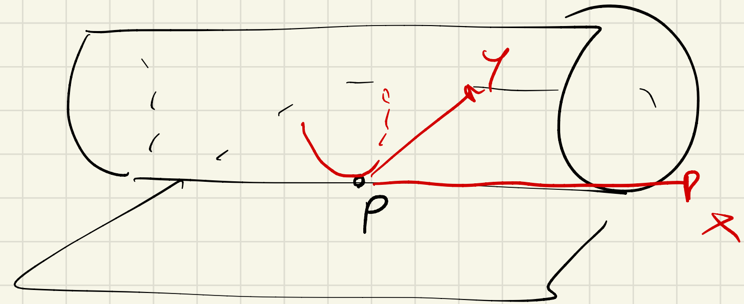
$$\underset{\text{scalare}}{R} = \underset{\text{gaussiana}}{2K}$$

$$f(x,y) = {}^t x \cdot \mathrm{Hess}\, f \cdot y + o(x^2+y^2)$$

$$\boxed{K = \det \mathrm{Hess}\, f}$$



$K=0$



$$f(x,y) = \lambda y^2$$

$$\mathrm{Hess}\, f = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$$